# 1 Random Walk

## 1.1 Symmetric Random Walk

### 1.1.1 Definition

Starting from coin toss,
def:

$$X_j = 1 \qquad if \;\; w_j = H$$
$$= -1 \qquad if \;\; w_j = T$$

Let $M_k = \sum_{j=1}^{k} X_j$, $M_k$ is symmetric random walk, and its distribution is binomial.

### 1.1.2 Expectation and Variance

$E(X_j) = 0$, $Var(X_j) = 1$,
$Var(M_k - M_l) = \sum_{i=l}^{k} Var(X_i) = k - l$;

### 1.1.3 Martingale Property

$E(M_l|F_k)$
$= E((M_l - M_k + M_k)|F_k)$
$= E((M_l - M_k)|F_k) + E(M_k|F_k)$
$= 0 + M_k = M_k$

## 1.2 Scaled Random Walk

### 1.2.1 Scaled Symmetric Random Walk Definition

$$W^{(n)}(t) = 1/\sqrt{n} M_{nt}$$
$$= 1/\sqrt{n} \sum_{i=1}^{nt} X^{(n)}(t)$$
$$= \sum_{i=1}^{nt} 1/\sqrt{n} X^{(n)}(t)$$

### 1.2.2 Expectation and Variance

$$E(W^{(n)}(t) - W^{(n)}(s)) = 0$$
$$Var(W^{(n)}(t) - W^{(n)}(s)) = n(t-s)Var(1/\sqrt{n} X^{(n)}(t-s)) = t - s$$

## 1.3 Brownian Motion

### 1.3.1 Limit of Scaled Symmetric Random Walk: Brownian Motion

When $n->$ inf, based on Central Limit Theorem
$W^{(n)}(t)$ starting from 0 follows a normal distribution N(0, t)
We view $W^{(n)}(t)$ as a Brownian motion

### 1.3.2 Quadratic variation of Brownian motion

Given a Brownian motion $W_{t_j}$ Define

$$
\begin{aligned}
Q &= \sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2 \\
E(Q) &= \sum_{j=0}^{n-1} E((W_{t_{j+1}} - W_{t_j})^2) \\
&= \sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= T
\end{aligned}
$$

$$
\begin{aligned}
Var(Q) &= \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2 \\
<= 2|C|T(where||C|| = max|t_{j+1} - t_j|) \\
When|C| = 0, \\
Var(Q) = 0
\end{aligned}
$$

This is simply written in differential form $dWdW = dt$

### 1.3.3 From Scaled Asymmetric Random Walk to Geometric Brownian motion

Consider a scaled asymmetric random walk with factor of $\sigma$ and drift $\alpha$

$$
W^{(n)}(t) = \sigma(\frac{1}{\sqrt{n}} M_{nt}) + \alpha t
$$

Construct new random variable which satisfies $\delta S/S = W^{(n)}(t)$ , which is equivalent to

$$
\begin{aligned}
S+ &= S(1 + \alpha/n + \sigma/\sqrt{n}) \\
S- &= S(1 + \alpha/n - \sigma/\sqrt{n})
\end{aligned}
$$

Assuming number of heads is $H_{nt}$, number of tails is $T_{nt}$ Then

$$S_n(t) = S_n(0)(1 + \alpha/n + \sigma/\sqrt{n})^{(H_{nt})}(1 + \alpha/n - \sigma/\sqrt{n})^{(T_{nt})}$$

$$\begin{aligned}
log(S_n(t)) &= log(S_n(0)) + (H_{nt})log(1 + \alpha/n + \sigma/\sqrt{n}) + (T_{nt})log(1 + \alpha/n - \sigma/\sqrt{n}) \\
&= log(S_n(0)) + (H_{nt})(\alpha/n + \sigma/\sqrt{n} - 1/2(\alpha/n + \sigma/\sqrt{n})^2) \\
&+ (T_{nt})(\alpha/n - \sigma/\sqrt{n} - 1/2(\alpha/n - \sigma/\sqrt{n})^2) \\
&= log(S_n(0)) + (H_{nt} + T_{nt})(\alpha/n - \sigma^2/n + O(n^{-3/2})) + (H_{nt} - T_{nt})(\sigma/\sqrt{n}) \\
&= log(S_n(0)) + nt(\alpha/n - 1/2\sigma^2/n) + M_{nt}\sigma/\sqrt{n} \\
&= log(S_n(0)) + (\alpha - \frac{1}{2}\sigma^2)t + W^n(t)\sigma
\end{aligned}$$

Then we can prove when $n- > \infty$ The distribution of $S_n(t)$ converges to the distribution of

$$S(t) = S(0)exp((\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t))$$

Where W(t) is a normal random variable with mean 0 and variance t.