1 Derivation of Capital Asset Pricing Model

Given an asset with return R_i, the basic pricing equation for return states

$$1 = E(mR_i) \tag{1}$$

This means if we invest using one dollar, the expectation of discount return should be equal to one dollar. In other words, it is equal to what we start with to prevent arbitrage. Applying the covariance decomposition

$$1 = E(m)E(R_i) + cov(m, R_i)$$

If we let $R_i = R_f$ in Eq.1, where R_f is the risk-free interest rate. Because R_f is deterministic, we have $R_f = \frac{1}{E(m)}$. Substituting the R_f into the equation above yields

$$1 = \frac{E(R_i)}{R_f} + cov(R_i, m)$$

After a little algebra

$$\begin{aligned} R_f &= E(R_i) + R_f cov(R_i, m) \\ E(R_i) = R_f - R_f cov(m, R_i) &= R_f - \frac{1}{E(m)}\rho(m, R_i)\sigma(m)\sigma(R_i) \end{aligned}$$

Because $\rho(m, R_i) <= 1$, so

$$|E(R_i) - R_f| \leq \frac{\sigma(m)}{E(m)}\sigma(R_i)$$

We define market portfolio return R_{mf}: $\rho(m, R_{mf}) = -1$, then

$$E(R_{mf}) = R_f - \frac{\sigma(m)}{E(m)}\sigma(R_{mf})$$

This is called the market front line with slope equal to $\frac{\sigma(m)}{E(m)}$.And

$$-\frac{\sigma(m)}{E(m)}\sigma(R_{mf}) = E(R_{mf}) - R_f \tag{2}$$

$$E(R_i) = R_f - \frac{cov(R_i, m)}{E(m)} = R_f - \frac{cov(R_i, m)}{var(m)}\frac{var(m)}{E(m)}$$

When $\rho(m, R_{mf}) = 1$, let $m = a + bR_{mf}$, then

$$\begin{aligned} E(R_i) &= R_f - \frac{bcov(R_i, R_{mf})}{b^2 var R_{mf}}\frac{\sigma(m)b\sigma(R_{mf})}{E(m)} \\ &= R_f - \frac{cov(R_i, R_{mf})}{var R_{mf}}\frac{\sigma(m)\sigma(R_{mf})}{E(m)} \end{aligned}$$

Substituting Eq.2 for the above expression, we have

$$
\begin{aligned}
E(R_i) &= R_f + \frac{cov(R_i, R_{mf})}{varR_{mf}}(E(R_{mf}) - R_f) \\
&= R_f + \beta(E(R_{mf}) - R_f)
\end{aligned}
$$

With $\beta = \frac{cov(R_i, R_{mf})}{varR_{mf}}$.