1 Ito Integral

a. Ito Integral Definition for simple integrand
Given t_0, t_1 t_n and Δt is a constant in between any $[t_k, t_{k+1}]$

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

We can also rewrite $I(t) = \int_0^t \Delta(u)dW(u)$.

b. Properties of Ito integral
1) Ito Integral is a martingale
2) Isometry
$EI^2(t) = E\int_0^t \Delta^2(u)du$

c. Ito integral definition for general integrand
Choose $\Delta_n(t)$ such that when $n- > \infty$
$lim_{n->\infty} E\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0$
Define Ito integral

$$\int_0^t \Delta(u)dW(u) = lim_{n->\infty} \int_0^t \Delta_n(u)dW(u)$$

2 Ito formula

a. Ito formula
Suppose $dX_t = udt + \sigma dB_t$
If g(t, X) is twice continuously differentiable $Y_t = g(t, X_t)$

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2$$

Where $(dX)^2 = d(X_t)d(X_t) = u^2(dt)^2 + 2u\sigma dtdB_t + \sigma^2(dB_t)^2$
$dt * dt = dt * dB_t = 0$, $dB_t dB_t = dt$
So $(dX_t)^2 = \sigma^2 dt$.

b. Example: Differentiation of Geometric Brownian motion
The geometric Brownian motion satisfies

$$S(t) = S(0)exp((\mu - \frac{1}{2}\sigma^2)t + \sigma dW(t))$$

Based on Ito's formula and chain rule

$$\begin{aligned}
&\frac{dS}{dt} = S(\mu - \frac{1}{2}\sigma^2) \\
&\frac{dS}{dW} = S(\sigma) \\
&\frac{d^2S}{dW^2} = \frac{d}{dW}(\frac{dS}{dW}) = \frac{d}{dW}(S(\sigma)) = \sigma\frac{dS}{dW} = \sigma^2 S
\end{aligned}$$

So

$$\begin{aligned}
dS &= S(\mu - \frac{1}{2}\sigma^2)dt + \sigma S dW(t) + \frac{1}{2}S\sigma^2 dt \\
&= S((\mu dt + \sigma dW(t))
\end{aligned}$$

c. Example: Integral of Geometric Brownian motion

The geometric Brownian motion satisfies $dN_t/N_t = rdt + \sigma dB_t$ to solve this we consider

$$\begin{aligned}
d(lnN_t) &= \frac{\partial lnN_t}{\partial N_t}dN_t + \frac{1}{2}\frac{\partial^2 lnN_t}{\partial N_t^2}(dN_t)^2 \\
&= \frac{1}{N_t}dN_t + \frac{1}{2}(-\frac{1}{N_t^2})(dN_t)^2 \\
(dN_t)^2 &= r^2N_t^2(dt)^2 + rN_t dt\sigma dB_t + \sigma^2 N_t^2 d^2B_t \\
&= 0 + 0 + \sigma^2 N_t^2 dt
\end{aligned}$$

So

$$\begin{aligned}
d(lnN_t) = \frac{1}{N_t}dN_t - \frac{1}{2}\sigma^2 dt = (r - \frac{1}{2}\sigma^2)dt + \sigma dB_t \\
ln(N_t/N_0) = (r - \frac{1}{2}\sigma^2)t + \sigma B_t \\
N_t = N_0 exp((r - \frac{1}{2}\sigma^2)t + \sigma B_t)
\end{aligned}$$