The objective of this article is to provide a comprehensive review of the Hall effect family. Grasping the intricate physics underlying the various manifestations of the Hall effect is no straightforward feat. In this regard, we present a step-by-step analysis. The structure of the entire article is as follows:
1) We commence by delving into the conventional theory of conductivity, spanning from classical principles to the realm of quantum mechanics.
2) Upon entering the quantum realm, we proceed to derive the current operator, encompassing two distinct contributions. The initial term corresponds to the gradient of the energy eigenvalue in k-space, impacting the conductivity of most metals. The subsequent term, connected to the Berry curvature, assumes significance in situations where time-reversal symmetry is violated.
3) It is worth noting that the Berry curvature-associated second term possesses a captivating property. The integration of the Berry curvature across the Brillouin zone, summed over all bands, results in an integer value of 2π.

1 Brief Summary of Conductance Theory

To elucidate conductivity, physicists have formulated multiple models, spanning from the classical and semiclassical to the quantum approaches. These models can be summarized as follows:
1) In classical model, the electrons are treated classically, and the movement is governed by the Newton's law and the forces on the electrons are described by electromagnetism . This model is good enough to explain the Ohm's law.
2) The semiclassical model views electrons as both particles and waves. Electron movement is likened to wavepacket propagation, with the electron's velocity representing the group velocity of the wave. This model leverages particle-wave duality and effectively accounts for conduction in metals.
3) In the quantum model, velocity is represented by the expectation value of the velocity operator within a given wavefunction. This theoretical approach is essential for deriving Hall conductance and comprehending the topological intricacies of Hall conductance.

2 Classical Conductance Theory Example: Hall Effect

We consider the electrons inside conductors. When we apply both an electric field E and a magnetic filed B, the electrons have the equation of motion following the Newtons' law

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - m\frac{\mathbf{v}}{\tau}$$

The first term in right hand side is the force by the electric field, and the second term is the force by the magnetic field. The third term electron collision by the ions. When collision happens, the momentum of the electron changes to zero within a certain mean free time τ. At the equilibrium states, we have $\frac{d\mathbf{v}}{dt} = 0$.

The velocity satisfies

$$\frac{e\tau}{m}\mathbf{v} \times \mathbf{B} + \mathbf{v} = -\frac{e\tau}{m}\mathbf{E} \tag{1}$$

As $\mathbf{v} = (v_x,\, v_y)$, so the above equation can be written as

$$v_x + \frac{e\tau}{m}v_y B = -\frac{e\tau}{m}E_x$$

The current density J is related to the velocity by

$$\mathbf{J} = -ne\mathbf{v}$$

So

$$j_x + \frac{e\tau B}{m}j_y = \frac{ne^2\tau}{m}E_x$$

We define the conductivity as

$$\mathbf{J} = \sigma\mathbf{E}$$

so $\sigma_{xx} = \frac{ne^2\tau}{m}$, $\sigma_{xy} = \frac{ne}{B}$.

3 Hall conductivity of 2D electrons

Solution to 2D electron system subject to a magnetic field
A Hamiltonian for 2D electrons in a magnetic field $A = xB\hat{y}$ is

$$H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2)$$

Because this Hamiltonian commutes with p_y, so they share the same eigenstates, therefore, we can write the solution for the Hamiltonian as

$$\psi_k(x, y) = e^{iky}f_k(x)$$

$$H\psi_k(x, y) = \frac{1}{2m}(p_x^2 + (\hbar k + eBx)^2)\psi_k(x, y) = H_k\psi_k(x, y)$$

$$H_k = \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + \frac{\hbar k}{eB})^2$$

This H_k is the Hamiltonian for a harmonic oscillator in the x direction, with the center displaced form the origin. The solution to H_k is very similar to harmonic oscillator. The energy eigenvalues are

$$E_n = \hbar\omega_B(n + \frac{1}{2})$$

where $\omega_B = \frac{eB}{m}$. And the eigenstate wavefunctions are

$$\psi_{n,k}(x,y) \propto e^{iky} H_n(x + \frac{\hbar k}{eB}) e^{-(x+\frac{\hbar k}{eB})^2 eB/2\hbar}$$

Adding an electric filed for 2D electron system subjected to a magnetic field

$$H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2) + eEx$$

Its solution is again similar to harmonic oscillator with additional shift

$$\psi(x,y) = \psi_{n,k}(x + mE/eB^2, y)$$

and the energies are

$$E_{n,k} = \hbar\omega_B(n + \frac{1}{2}) - eE(\frac{\hbar k}{eB} + \frac{eE}{m\omega_B^2}) + \frac{m}{2}\frac{E^2}{B^2}$$

Since we get the wavefunction and eigenenergy, there are two ways to find out the current. One way is to use the semiclassical approach. We can calculate group velocity given a wavevector k

$$v_y = \frac{1}{\hbar}\frac{\partial E_{n,k}}{\partial k} = -\frac{E}{B}$$

So we surprisingly see add an electric filed in x direction generates the movement in y! To find out the total current in y direction, we have to know the degeneracy, which means how many electrons are in the state with the momentum k to k + dk. In y direction, electrons are free particle with momentum k confined in a finite size L_y. So

$$\frac{dn}{dk} = \frac{L_y}{2\pi}$$

The total current is

$$I_y = e\frac{E}{B}\int \frac{dn}{dk} dk = \frac{eEL_y}{2\pi B}\int k$$

The range of k in the above integral is tricky. From the wavefunction, we see the center of the harmonic oscillator in x direction is $x = -k\hbar/eB$, while $0 \leq x \leq L_x$, then $-L_x eB/\hbar \leq k \leq 0$.

$$I_y = \frac{eEL_y}{2\pi B}\int_{-L_x eB/\hbar}^{0} k = \frac{e^2}{h}EA$$

The second way is purely quantum approach. We need the following steps in order to derive the expression of the current. 1) In quantum mechanics, we represent every physical quantity using an operator. So the first step is to find out the expression of velocity operator.
2) The current is the expectation value of its operator, therefore we need to know the wavefunction. Here we employ the time-dependent perturbation theory in

order to know how the wavefunction changes overtime.
3) Thirdly, we evaluate the expectation of the current operator then analyze why the conductivity is quantized.

Derivation of velocity operator

In quantum mechanics, the velocity operator is defined as

$$v = \frac{\partial H}{\partial p} = \frac{\partial H}{\hbar \partial k}$$

There exist several ways to understand this. First, we can recall the equation of motion in analytical mechanics. Given a Hamiltonian, the equation of motion is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}$$

Another way is using Heisenburg equation of motion, which is the counterpart of analytical mechanics' equation of motion in quantum mechanics. The velocity operator given by Heisenburg equation of motion is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{i}{\hbar}[H, \mathbf{r}]$$

In momentum space, it becomes

$$\mathbf{v}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{i}{\hbar}[H, \mathbf{r}] e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{\hbar} \nabla_{\mathbf{k}} H(\mathbf{k}, t)$$

Wavefunction subject to adiabatic evolution

The wave function is subject to the time-dependent Schrodinger equation.

$$i\hbar \partial_t |\Psi(t) >= H(t)|\Psi(t) >$$

Suppose when $t = t_0$, the instantaneous eigenstates are $|u_n(\mathbf{k}, t) >$, then the wavefunction can be written as linear combination of all instantaneous states with coefficients $a_n(t)$ times a time evolution factor.

$$|\Psi(t) >= \sum_n exp(\frac{1}{i\hbar} \int_{t_0}^{t} dt^{'} E_n(t^{'})) a_n(t) |u_n(\mathbf{k}, t) >$$

Then we consider adiabatic approximation which means the vector $\mathbf{R}$(t) varies with time very slowly and apply the time-dependent perturbation theory. After a few steps we have

$$|\Psi(t) >= exp(-\frac{i}{\hbar} \int_{t_0}^{t} dt^{'} E_n(t^{'}))(|u_n(\mathbf{k}) > -i\hbar \sum_{n^{'} \neq n} |u_n^{'}(\mathbf{k}) > \frac{< u_n^{'}(\mathbf{k})|\frac{\partial}{\partial t}|u_n(\mathbf{k}) >}{E_n - E_{n'}})$$

The expectation value of the velocity

$$\bar{v}(\mathbf{k}, t) = \frac{1}{\hbar} \nabla_{\mathbf{k}} E_n(\mathbf{k}) - i \sum_{n^{'} \neq n} (< u_n|\frac{\partial H}{\partial \mathbf{k}}|u_n^{'} > \frac{< u_n^{'}|\frac{\partial}{\partial t}|u_n >}{E_n - E_{n'}} - c.c.)$$

Using the identity

$$< u_n|\nabla_{\mathbf{k}} H|u_m >= (E_n - E_m) < \nabla_{\mathbf{k}} u_n|u_m >$$

$$\bar{v}(\mathbf{k},t) = \frac{1}{\hbar}\nabla_{\mathbf{k}} E_n(\mathbf{k}) - i(< \frac{\partial u_n}{\partial \mathbf{k}}|\frac{\partial u_n}{\partial t} > - < \frac{\partial u_n}{\partial t}|\frac{\partial u_n}{\partial \mathbf{k}} >) \tag{2}$$

Where the second term is the **Berry phase**. The current operator is

$$j = -2e \sum_{allbands} \int_{BZ} \frac{dk}{2\pi} f(k) v(k)$$

The integral is taken over the first Brillouin zone denoted by BZ. We need to consider several cases to discuss the current.
1) If the system preservers time reversal symmetry and space reversal symmetry, then the second term in Eqn. 2 vanishes. In this case, if the bands are fully occupied, it is an insulator. If the band are not fully occupied, it is an conductor. The semiconductor is something between this two where the thermal excitation can promote the electron going from valence band to the conduction band so the not fully occupied conduction band contributes the current.
2) If all the bands are filled, but the system breaks time and space reversal symmetry, then first term vanishes, the second term is non-trivial. This is what contributes the current in quantum Hall effect.

Quantized current

When the first term of the velocity vanishes(in the case that all bands are fully occupied), the expression of velocity reduces to

$$\bar{v}(\mathbf{k},t) = -i(< \frac{\partial u_n}{\partial \mathbf{k}}|\frac{\partial u_n}{\partial t} > - < \frac{\partial u_n}{\partial t}|\frac{\partial u_n}{\partial \mathbf{k}} >) \tag{3}$$

Using the relationship

$$\partial_t = \partial_t \mathbf{k} \cdot \nabla_{\mathbf{k}} = -\frac{e}{\hbar}\mathbf{E} \times \Omega^n(\mathbf{k})$$

where

$$\Omega^n(\mathbf{k}) = \nabla_{\mathbf{k}} \times < u_n(\mathbf{k})|i\nabla_{\mathbf{k}}|u_n(\mathbf{k}) >$$

$$\mathbf{v_n}(\mathbf{k}) = -\frac{e}{\hbar}\mathbf{E} \times \Omega^n(\mathbf{k})$$

We plug the expression of v_n into the expression of current j, we have

$$\mathbf{j} = -2e \sum_{allbands} \int_{BZ} \frac{d\mathbf{k}}{2\pi} f(\mathbf{k}) \mathbf{v}_n(\mathbf{k}) = \frac{e^2}{h}\frac{1}{2\pi}\sum_n \int_{BZ} d\mathbf{k}\mathbf{E} \times \Omega^n(\mathbf{k})$$

Therefore the conductivity is

$$\sigma = \frac{e^2}{h}\frac{1}{2\pi}\sum_n \int_{BZ} d\mathbf{k}\Omega^n(\mathbf{k})$$

If we consider the case of 2 dimensional electron gas, then

$$\sigma = \frac{e^2}{h}\frac{1}{2\pi}\sum_n \int_{BZ} d\mathbf{k}\Omega^n(k_x, k_y)$$

Since the integral runs over the first Brillouin zone, and

$$\Omega^n(k_x, k_y) = \Omega^n(k_x + \pi, k_y) = \Omega^n(k_x, k_y + \pi)$$

Hence, the first Brillouin zone forms a closed torus. The integral over a closed torus gives an multiple integer of 2π. So

$$\sigma_H = \nu \frac{e^2}{h}$$

Where ν is an integer.