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1 Symmetric Matrix

Definition

$$A = A^T$$

Properties
1) We know the eigenvectors associated with distinct eigenvalues are linearly independent for all matrix. The eigenvectors associate with distinct eigenvalues (v_1, v_2, v_n)of a symmetric matrix are not only linearly independent but also orthogonal. So let V be the matrix whose columns are the eigenvectors of A, then $VV^T = I$, $V^{-1} = V^T$. If with same eigenvalues, then the eigenvector may not be orthogonal, we can do Gram-Schmit transformation to make it orthogonal.
2) The diagonal factorization of an symmetric matrix is

$$\begin{aligned} A =& VCV^T \\ =& AI \\ =& A\sum_i v_i v_i^T \\ =& \sum_i c_i v_i v_i^T \end{aligned}$$

3) Maximum value of As quadratic form

$$\begin{aligned} & x^T Ax \\ =& x^T \sum_i c_i v_i v_i^T x \\ =& \sum_i b^T V^T v_i v_i^T V b c_i \\ =& \sum_i b_i^2 c_i \\ <=& max(c_i) b^T b \\ =& max(c_i) x^T x \end{aligned}$$

2 Hermitian Matrix

A = A*(where A* is the complex conjugate of A)
1) The eigenvalues are real.

3 Orthogonal Matrix

Definition
$A{-}1 = A^T$
Intuition
Orthogonal matrix arise from dot product. Consider vector u , and a matrix Q. When we apply the matrix Q to v, we get $v^{'} = Qv$. We would like to have the dot product preserved, namely

$$v^T v = v^{'T} v^{'} = (Qv)^T (Qv) = v^T Q^T Q v$$

So $Q^T Q = 1$, $Q^T = Q^{-1}$.
Properties
Transformations by orthogonal matrices are called orthogonal transformations. One property of orthogonal transformation is it preserves the angles between the two vectors. If Q is an orthogonal matrix($Q^T Q = I$), then for vectors x and y, we have

$$< Qx, Qy > = < xQ >^T < Qy > = x^T Q^T Q y = x^T y = < x, y >$$

so the new angle between Qx and Qy is

$$arccos(\frac{< Qx, Qy >}{||Qx||_2 ||Qy||_2}) = arccos(\frac{< x, y >}{||x||_2 ||y||_2})$$

The new angle is equal to the old angle of x and y. Take an two dimension vector as an example. If Q is

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

And $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$, if $< x, y >= 0$. Then

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q_{11}x_1 + q_{12}x_2 \\ q_{21}x_1 + q_{22}x_2 \end{pmatrix}$$

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} q_{11}y_1 + q_{12}y_2 \\ q_{21}y_1 + q_{22}y_2 \end{pmatrix}$$

Then

$$\begin{pmatrix} q_{11}x_1 + q_{12}x_2 & q_{21}x_1 + q_{22}x_2 \end{pmatrix} \begin{pmatrix} q_{11}y_1 + q_{12}y_2 \\ q_{21}y_1 + q_{22}y_2 \end{pmatrix}$$

$=q_{11}^2 x_1 y_1 + q_{11}q_{12}(x_1 y_2 + x_2 y_1) + q_{12}^2 x_2 y_2 + q_{21}^2 x_1 y_1 + q_{21}q_{22}(x_1 y_2 + x_2 y_1) + q_{22}^2 x_2 y_2$
$=x_1 y_1 (q_{11}^2 + q_{21}^2) + x_2 y_2 (q_{12}^2 + q_{22}^2) + (q_{11}q_{12} + q_{21}q_{22})(x_1 y_2 + x_2 y_1)$
$=x_1 y_1 * 1 + x_2 y_2 * 1 + 0(x_1 y_2 + x_2 y_1)$
$=x_1 y_1 + x_2 y_2$

Examples
1) Rotation Matrix

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

2) Reflection Matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4 Idempotent Matrix

Definition
$A^2 = A$
Properties
1) Its eigenvalues are either 0 or 1. Because the eigenvalues of A^2 are the squares of the eigenvalues of A. 2) Any vector in the columns space of an idempotent matrix A is an eigenvector of A 3) The number of eigenvalues that are 1 is the rank of an idempotent matrix. $tr(A) = rank(A)$

5 Symmetric Positive Definite

Definition
A symmetric positive definite matrix satisfies for any non-zero vector x, $x^T Ax > 0$
Properties 1) Positive definite matrix is non-singular.
Proof: If A is singular, it means there is a non-zero vector x so that Ax=0. Therefore $x^T Ax = 0$, which is a contradiction.
2) All the eigenvalues are positive.
3) Its leading principal minors are all positive.
4) It has a unique Cholesky decomposition.