1 Linear Regression

1.1 Linear regression and Least Square Solution

$$Y = X\boldsymbol{\beta} + \epsilon$$

Where Y is a n × 1 matrix, X is a n × k matrix, beta is k × 1 vector and ϵ is nx1 vector with ϵ_i begin iid with normal distribution.

Assumptions

1. Linear
2. X matrix has full rank. In other words, no multicollinearity.
2. error term has zero mean $E[\epsilon|X] = 0$
3. Homescedasticity or equal variance of ϵ. In other words, no autocorrelation between disturbances.$cov(\epsilon_i, \epsilon_j) = 0$.
6. Number of obsearvations n must be greater than the number of parameters.

Least Square Solution

The cost function is given by

$$f(\boldsymbol{\beta}) = ||Y - X\beta||^2 = (Y - X\beta)^T(Y - X\beta) \quad = Y^TY - Y^TX\beta - \beta^TX^TY + \beta^TX^TX\beta$$

Since third term are scalar,

$$\beta^TX^TY = (\beta^TX^TY)^T = Y^TX\beta$$

$$f(\beta) = Y^TY - 2Y^TX\beta - \beta^TX^TX\beta = Y^TY - 2(X^TY)^T\beta + \beta^TX^TX\beta$$

The first term is a constant and its derivative is zero.

The deriviative of 2nd term

Consider the derivative of $\alpha^T\beta$ with respect to β.

$$\boldsymbol{\alpha}^T\boldsymbol{\beta} = \Sigma\alpha_i\beta_i$$
$$\frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_i} = \alpha_i$$

Write the derivative in matrix form

$$\begin{pmatrix} \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_1} \\ \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_2} \\ \cdots \\ \frac{\partial\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\partial\beta_3} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_p \end{pmatrix}$$

So if we let $\alpha = X^TY$, we have

$$\frac{\partial 2(X^TY)^T\beta}{\partial\beta} = 2X^TY$$

The derivative of 3rd term

let $A = X^TX$,

$$\beta^TX^TX\beta = \beta^T\begin{pmatrix} \Sigma_iA_{1k}\beta_k \\ \Sigma_iA_{2k}\beta_k \\ \cdots \\ \Sigma_kA_{pk}\beta_k \end{pmatrix} = \Sigma_j\beta_j(\Sigma_kA_{jk}\beta_k)$$

To calculate the derivative of $f(\beta)$, we note there are only 3 cases that the derivative does not vanish
1) l = j = k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = 2A_{ll}\beta_l$$

2) l=j, j $\neq$ k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = \Sigma_{k,k\neq l} A_{lk}\beta_k$$

3) l=k, j $\neq$ k

$$\frac{f(\boldsymbol{\beta})}{\partial \beta_l} = \Sigma_{j,j\neq l} A_{jl}\beta_j = \Sigma_{j,j\neq l} A^T_{lj}\beta j$$

Therefore

$$\begin{aligned}\frac{f(\boldsymbol{\beta})}{\partial \beta_l} &= A_{ll}\beta_l + \Sigma_{k,k\neq l} A_{lk}\beta_k + A_{ll}\beta_l + \Sigma_{j,j\neq l} A^T_{lj}\beta j \\ &= \Sigma_k A_{lk}\beta_k + \Sigma_j A^T_{lj}\beta j\end{aligned}$$

The first term is the lth row of vector $A\beta = X^T X\beta$, and the 2nd term is the lth row of vector$A^T\beta = X^T X\beta$. So we put the whole derivative in matrix form

$$\frac{f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2X^T Y + 2X^T X\beta$$

which is a px1 vector with each row corresponding to the derivative with respect to β_i letting the derivative equal to zero yields the **normal equation** and the estimation of β
Normal equation

$$(X^T X)\hat{\beta} = X^T Y$$

Estimator of β

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Least Square Estimator for Simple Linear Regression

$$y = \beta_0 + \beta_1 X + \epsilon$$

$$\begin{aligned}&\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ =&(X^T X)^{-1} X^T Y \\ =&\left(\begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_n \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix} \\ =&\frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{pmatrix} \Sigma_i x_i^2 & -\Sigma_i x_i \\ -\Sigma_i x_i & n \end{pmatrix} \begin{pmatrix} \Sigma_i y_i \\ -\Sigma x_i y_i \end{pmatrix}\end{aligned}$$

So

$$\beta_0 = \frac{\Sigma x_i^2 \Sigma y_i - \Sigma x_i (\Sigma x_i y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2} \tag{1}$$

$$\beta_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} \tag{2}$$

β_1 can also be written using the covariance

$$\beta_1 = \frac{\sum_i^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_i^n (x_i - \bar{x})(x_i - \bar{x})} \tag{3}$$

And it is easy to show

$$\begin{aligned}
\beta_2 &= \frac{\sum_i^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_i^n (x_i - \bar{x})(x_i - \bar{x})} \\
&= \frac{\sum_i^n (x_i y_i - \bar{x} y_i - x_i \bar{y} + \bar{x}\bar{y})}{\sum_i^n (x_i^2 - 2\bar{x}x_i + (\bar{x})^2)} \\
&= \frac{\sum_i^n x_i y_i - \sum_i^n \bar{x} y_i - \sum_i^n x_i \bar{y} + \sum_i^n \bar{x}\bar{y}}{\sum_i^n x_i^2 - \sum_i^n 2\bar{x}x_i + \sum_i^n (\bar{x})^2} \\
&= \frac{\sum_i^n x_i y_i - (\frac{1}{n}\sum_j^n x_j)(\sum_i^n y_i) - (\sum_i^n x_i)(\frac{1}{n}\sum_j^n y_j) + \sum_i^n (\frac{1}{n}\sum_j^n x_i)(\frac{1}{n}\sum_k^n y_i)}{\sum_i^n x_i^2 - \sum_i^n 2(\frac{1}{n}\sum_j^n x_j)x_i + \sum_i^n (\frac{1}{n}\sum_j^n x_j)^2} \\
&= \frac{\sum_i^n x_i y_i - (\frac{1}{n}\sum_j^n x_j)(\sum_i^n y_i) - (\sum_i^n x_i)(\frac{1}{n}\sum_j^n y_j) + n(\frac{1}{n}\sum_j^n x_i)(\frac{1}{n}\sum_k^n y_k)}{\sum_i^n x_i^2 - \sum_i^n 2(\frac{1}{n}\sum_j^n x_j)x_i + n(\frac{1}{n}\sum_j^n x_j)^2} \\
&= \frac{\sum_i^n x_i y_i - \frac{1}{n}(\sum_i^n x_i)(\sum_j^n y_j) - \frac{1}{n}(\sum_i^n x_i)(\sum_j^n y_j) + \frac{1}{n}(\sum_j^n x_i)(\sum_k^n y_k)}{\sum_i^n x_i^2 - \frac{2}{n}(\sum_j^n x_j)(\sum_i^n x_i) + \frac{1}{n}(\sum_j^n x_j)^2} \\
&= \frac{\sum_i^n x_i y_i - \frac{1}{n}(\sum_i^n x_i)(\sum_j^n y_j)}{\sum_i^n x_i^2 - \frac{1}{n}(\sum_j^n x_j)(\sum_i^n x_i)} \\
&= \frac{n\sum_i^n x_i y_i - (\sum_i^n x_i)(\sum_j^n y_j)}{n\sum_i^n x_i^2 - (\sum_j^n x_j)(\sum_i^n x_i)} \\
&= \frac{n\sum x_i y_i - (\sum x_i)(\sum_j^n y_j)}{n\sum x_i^2 - (\sum x_i)^2}
\end{aligned}$$

which is the same as Eq.2. We can interpret β as ratio of the covariance of x and y to the variance of x.

1.2 Projection matrix

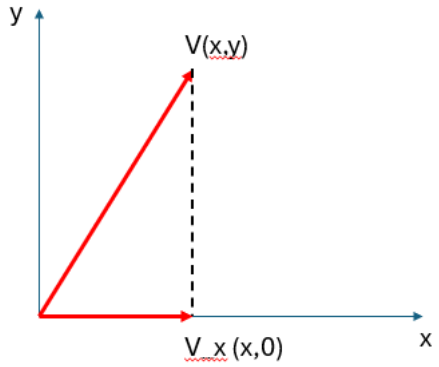
Given $\hat{\beta} = (X^T X)^{-1} X^T Y$, we have the predictor value of $y = X\beta$

$$\hat{y} = X(X^T X)^{-1} X^T y$$

The matrix $P = X(X^T X)^{-1} X^T$ is a projection matrix. It projects the vector of y into the column space of X.

Understand the word projection

Let us understand this first through geometry point of view. Consider a vector on 2 dimensional space, $V_1 = (x_1, y_1)^T$, where x_1 and y_1 are the x and y component, respectively. If we project the vector V into x-line, then apparently we get $V_x = (x_1, 0)^T$, see graph below.



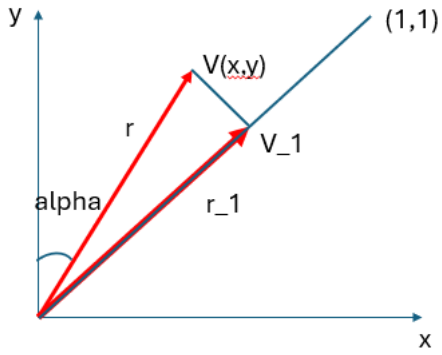
If we have a vector that is along the x axis

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The projection matrix of a vector into x line is

$$\begin{aligned} P_x &= x(x^T x)^{-1} x^T \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Applying this projection matrix to any 2 dimensional vector V gives $(V_x, 0)^T$. So it projects the vector into x line. Let us take another example. Imagine V_1 is vector if we project V onto the line that has 45 degree angle with x axis. See below.



In order to calculate V_1, we see

$$r_1 = rcos(\pi/4 - alpha) = r(\frac{\sqrt{2}}{2}\frac{y}{r} + \frac{\sqrt{2}}{2}\frac{x}{r}) = \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}x$$

$$V_{1x} = r_1 cos(\pi/4) = \frac{x+y}{2}$$
$$V_{1y} = r_1 sin(\pi/4) = \frac{x+y}{2}$$

After we understand this using geometry point of view, we can workout from algebra point of view. The vector we want to project onto is

$$i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The projection matrix of a vector into x line is

$$\begin{aligned} P_x &= x(x^T x)^{-1} x^T \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Therefore we easily see

$$V_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x+y) \end{pmatrix}$$

which is the same as what we get based on geometry. For n dimensional vector y, if our X matrix has rank of k, then the projection matrix P projects the vector

y into k dimensional hyperplane. For example, if we define

$$i_N = \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}$$

The projection matrix P is

$$P = i\frac{1}{N}i^T = \frac{1}{N}\begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ ... & ... & ... & ... \\ 1 & 1 & ... & 1 \end{pmatrix}$$

Projection matrix into null space
If P is a projection matrix, the matrix $I - P$ is also a projection matrix. In linear regression model

$$y = X\beta + \epsilon$$
$$P = X(X^TX)^{-1}X^T$$

Define residual vector $\hat{\epsilon}$

$$\hat{\epsilon} = (I - P)y = (I - X(X^TX)^{-1}X^T)y$$

And it is easy to show $\hat{e}$ and X are orthogonal.

$$X^T\hat{\epsilon} = X^T(I - P)y = X^T(I - X(X^TX)^{-1}X^T)y = (X^T - X^TX(X^TX)^{-1}X^T)y = 0y = 0$$

For the above example, we define $M = I - \frac{1}{N}ii^T$, and My express the mean deviations of a vector.
Idempotent property of projection matrix
Consider the previous example that we project a vector V onto x axis, how about we do this projection twice, we would end up the same vector V_x. Using a little matrix algebra, it is easy to prove that for any project matrix P, we have $PP = P$.

1.3 Partitioned Regression and Regression

$$y = X\boldsymbol{\beta} + \epsilon = X_1\beta_1 + X_2\beta_2 + \epsilon$$

The normal equation is

$$\begin{pmatrix} X_1^TX_1 & X_1^TX_2 \\ X_2^TX_1 & X_2^TX_2 \end{pmatrix}\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1^Ty \\ X_2^Ty \end{pmatrix}$$

If X1 and X2 are orthogonal, namely, $X_1^TX_2 = 0$, then

$$\hat{\boldsymbol{\beta}}_{\mathbf{1}} = (X_1^TX_1)^{-1}X_1^Ty$$
$$\hat{\boldsymbol{\beta}}_{\mathbf{2}} = (X_2^TX_2)^{-1}X_2^Ty$$

If X1 and X2 are not orthogonal, we can solve for β_2 in the above normal equation set and get β_2

$$\begin{aligned}\hat{\beta}_2 &= [X_2^T(I - X_1(X_1^T X_1)^{-1}X_1^T)X_2]^{-1}[X_2(I - X_1(X_1^T X_1)^{-1}X_1^T)y] \\ &= (X_2^T M_1 X_2)^{-1}(X_2^T M_1 y)\end{aligned}$$

Given the fact that M_1 is symmetrical and idempotent, we can rewrite the above expression

$$\begin{aligned}\hat{\beta}_2 &= (X_2^T M_1 M_1 X_2)^{-1}(X_2^T M_1 M_1 y) \\ &= (X_2^T M_1^T M_1 X_2)^{-1}(X_2^T M_1^T M_1 y) \\ &= ((M_1 X_2)^T M_1 X_2)^{-1}((M_1 X_2)^T M_1 y)\end{aligned} \tag{4}$$

The above uses the property that $M_1^T = M_1$ and $M_1 M_1 = M_1$
The $\hat{\beta}_2$ is also the solution of

$$M_1 Y = M_1 X_2 \beta_2 + \epsilon$$

where $M_1 y$ is the residual of y regressed on X_1 and $M_1 X_2$ is the residual of X_2 regressed on X_1. For example, in simple linear regression

$$Y = \beta_0 + x\beta_1$$

Where $X_1 = 1_N$, so its projection matrix is $i\frac{1}{N}i^T$, and the corresponding M matrix is $I - \frac{1}{N}ii^T$. We tries to calculate β using partition regression.

$$MY = (I - \frac{1}{N}ii^T)Y = Y - \bar{Y} \quad MX = (I - \frac{1}{N}ii^T)Y = X - \bar{X}$$

Then

$$\beta_1 = ((MX)^T(MX))^{-1}((MX)^T(MY)) = \frac{\sum_i^N (x_i - \bar{x})(y_i - \bar{y})}{\sum_i^N (x_i - \bar{x})^2} \tag{5}$$

which is the same as Eq.3

1.4 Variance componet identity

If we define our mean projection matrix P

$$P = i\frac{1}{N}i^T$$

and silimarly we define mean deviation project matrix

$$M = I - P = i\frac{1}{N}i^T$$

We have

$$y = \hat{y} + \hat{\epsilon} = X\hat{\beta} + \hat{\epsilon}$$

Multiplying M matrix on the left, we have

$$My = MX\hat{\beta} + M\hat{\epsilon} = MX\hat{\beta} + \hat{\epsilon}$$

$$\begin{aligned}(My)^2 &= (MX\hat{\beta} + \hat{\epsilon})^T(MX\hat{\beta} + \hat{\epsilon}) \\ &= (\beta^T X^T M^T + \hat{\epsilon}^T)(MX\hat{\beta} + \hat{\epsilon}) \\ &= (MX\hat{\beta})^2 + \beta^T X^T M^T \hat{\epsilon} + (\beta^T X^T M^T \hat{\epsilon})^T + (\hat{\epsilon})^2\end{aligned}$$

The 2nd and 3rd terms are zero because that 1)$\hat{\epsilon}$ has zero mean, so $M^T\hat{\epsilon} = M\hat{\epsilon} = \hat{\epsilon}$ and 2) $X^T\hat{\epsilon} = 0$, so

$$(My)^2 = (MX\hat{\beta})^2 + (\hat{\epsilon})^2$$

Rewriting the above equation using summation, we have

$$\sum_i (y_i - \bar{y})^2 = \sum_i^N (\bar{y}_i - \bar{\hat{y}})^2 + \sum_i (y_i - \hat{y})^2$$

Define

$$\begin{aligned}SST &= \sum_i (y_i - \bar{y})^2 \\ SSR &= \sum_i (\bar{y}_i - \bar{\hat{y}})^2 \\ SSE &= \sum_i (y_i - \hat{y})^2\end{aligned}$$

Then we have

$$SST = SSR + SSE$$

1.5 Variance of $\hat{\beta}$ and σ^2 estimation

$$\begin{aligned}Var(\hat{\beta}) = Var((X^TX)^{-1}X^T\epsilon) &= (X^TX)^{-1}X^T Var(\epsilon)((X^TX)^{-1}X^T)^T \\ &= \sigma^2(X^TX)^{-1}X^TX(X^TX)^{-1} = \sigma^2(X^TX)^{-1}\end{aligned}$$

The above derivation use the fact that ϵ has a normal distribution with mean 0 and variance σ^2. For simple linear regression

$$Var(\hat{\beta}) = \frac{\sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{pmatrix} \Sigma_i x_i^2 & -\Sigma_i x_i \\ -\Sigma_i x_i & n \end{pmatrix}$$

$$Var(\hat{\beta}_0) = \frac{\Sigma x_i^2 \sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$Var(\hat{\beta}_1) = \frac{n\sigma^2}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

Try

$$\begin{aligned}\Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \Sigma_i(x_i^2 - 2(\Sigma_j \frac{x_j}{n})x_i + \frac{(\Sigma_j x_j)^2}{n^2}) \\ = \Sigma_i x_i^2 - \frac{2}{n}(\Sigma_i x_i)^2 + \frac{(\Sigma_i x_i)^2}{n} = \Sigma_i x_i^2 - \frac{1}{n}(\Sigma x_i)^2\end{aligned}$$

So

$$Var(\hat{\beta}_0) = \frac{\Sigma x_i^2 \sigma^2}{n\Sigma(x_i - \bar{x})^2}$$

$$Var(\hat{\beta}_1) = \frac{n\sigma^2}{n\Sigma(x_i - \bar{x})^2} = \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}$$

$$\begin{aligned}SSE &= \Sigma_i(y - \hat{y}_i)^2 \\ &= (Y - X\beta)^T(Y - X\beta) \\ &= (Y - X(X^TX)^{-1}X^TY)^T(Y - X(XTX) - 1X^TY) \\ &= (Y - PY)^T(Y - PY) \\ &= Y^T(1-P)^T(1-P)Y = Y^T(1-P)Y \\ &= (X\beta + \epsilon)^T(1-P)(X\beta + \epsilon) \\ &= \beta^T X^T(1-P)X\beta + 2\beta^T X^T X^T(I-P)\epsilon + \epsilon^T(I-H)\epsilon\end{aligned}$$

$$E[SSE] = E[\epsilon^T(I-P)\epsilon] = E[\epsilon^T\epsilon]trace(I-H) = \sigma^2(n-k)$$

We obtain the unbiased estimator of σ^2

$$\hat{\sigma}^2 = \frac{SSE}{n-k}$$

Therefore the estimator of variance of β

$$\hat{V}ar(\hat{\beta}_i) = \hat{\sigma}^2(X^TX)^{-1}_{ii}$$

and the stanard error of β_i is

$$SE(\hat{\beta}_i) = \sqrt{\hat{\sigma}^2(X^TX)^{-1}_{ii}}$$

2 Properties of Least Square Estimators

When we have a estimator, we need to evaluate how good our estimator is? A few questions we can ask is 1): how far is the value of our estimator away from

the true value, even in the ideal case when the sample size is inifinite? 2) when 1) is true, with finite sample size, does the value of our estimator approach to the true value as the sample size increase? In other words, does the estimator converge to the true value as sample size goes to inifinity? 3) when 1) and 2) is true, as the sample size increase, how fast does our estimator converges to true value? 4) with 1) 2) and 3), what is the asymptotic distribution of the estimator? If the distribution is normal, it can be used to do interval estimation such as confidence interval. The 1st question defines unbiasness, the 2nd one defines consistency, and the 3rd one defines efficiency.

2.1 Unbiasness

Unbiased

$$\begin{aligned}\hat{\beta} &= (X^T X)^{-1} X^T Y \\ &= (X^T X)^{-1} X^T (X\beta + \epsilon) \\ &= (X^T X)^{-1} X^T X\beta + (X^T X)^{-1} X^T \epsilon \\ &= \beta + (X^T X)^{-1} X^T \epsilon\end{aligned}$$

Then the expectation of $\hat{\beta}$ condition on X is

$$E[\hat{\beta}|X] = \beta + (X^T X)^{-1} X^T E(\epsilon|X)$$
$$t$$

The last term is zero by assuption of linear regression. So

$$E[\hat{\beta}] = \beta$$

The expectation of the estimator is the same as true value, this is called **unbiased**.
Bias due to omission of relevant variables
Suppose we have a model

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

If we regression y on X_1 only, our estimator is

$$\hat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T y = \beta_1 + (X_1^T X_1)^{-1} X_1^T X_2\beta_2 + (X_1^T X_1)^{-1} X_1^T \epsilon$$

On the second term, we see unless 1)X_1 and X_2 are orthogonal, or 2)β_2 =0, β_1 is biased.

2.2 Consistency

The unbiasness gives us a metric of measuring how good our esitimator is, from population perspetive. In reality, as our sample size is finite, we need

ask ourselves does our estimator converges to true value when sample size is sufficiently large. We know

$$\hat{\beta} = \beta + (X^T X)^{-1} X^T \epsilon$$

$$\begin{aligned}
X^T X &= \Sigma_{i=1}^N \begin{pmatrix} x_{1i}^T x_{i1} & x_{1i}^T x_{i2} & ... & x_{1i}^T x_{ik} \\ ... & ... & ... & \\ x_{ki}^T x_{i1} & x_{ki}^T x_{k2} & ... & x_{ki}^T x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N \begin{pmatrix} x_{i1} x_{i1} & x_{i1} x_{i2} & ... & x_{i1} x_{ik} \\ ... & ... & ... & \\ x_{ik} x_{i1} & x_{ik} x_{k2} & ... & x_{ik} x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N \begin{pmatrix} x_{i1} \\ ... \\ x_{ik} \end{pmatrix} \begin{pmatrix} x_{i1} & ... & ...x_{ik} \end{pmatrix} \\
&= \Sigma_{i=1}^N X_i X_i^T
\end{aligned}$$

$$\begin{aligned}
\hat{\beta} &= \beta + (X^T X)^{-1} X^T \epsilon \\
&= \beta + (\Sigma_{i=1}^N X_i X_i^T)^{-1} X^T \epsilon \\
&= \beta + (\Sigma_{i=1}^N \frac{1}{N} X_i^T X_i)^{-1} (\frac{X^T \epsilon}{N})
\end{aligned}$$

If X_is are iid, then by law of large numbers

$$\Sigma_{i=1} \frac{1}{n} X_i^T X_i$$

converges to Q in probability.

$$\begin{aligned}
&\frac{X^T \epsilon}{N} \\
=&\begin{pmatrix} \frac{1}{N} \Sigma_1^N x_{i1} \epsilon_i \\ \frac{1}{N} \Sigma_1^N x_{i2} \epsilon_i \\ \frac{1}{N} \Sigma_1^N x_{i3} \epsilon_i \\ ... \\ \frac{1}{N} \Sigma_1^N x_{ik} \epsilon_i \end{pmatrix} \\
=&\frac{1}{N} \Sigma_{i=1}^N X_i \epsilon_i = \bar{w}
\end{aligned}$$

Where $\bar{w}$ is a k $\times$ 1 vector. To see the asymptotical behavior of w, we consider its mean and asymptotical varance. The mean is

$$E[w_i] = E_X[E[w_i|x_i]] = E_X[X_i E[\epsilon|X_i]] = 0$$

$$Var[\bar{w}] = E[Var[\bar{w}|X]] + Var[E[\bar{w}|X]] = E[Var[\bar{w}|X]] + 0 = E[Var[\bar{w}|X]]$$

$$Var[\bar{w}|X] = E[\bar{w}\bar{w}^T|X] = \frac{1}{n} X^T E[\epsilon\epsilon^T] X \frac{1}{n} = \frac{\sigma^2}{n} \frac{X^T X}{n}$$

$$E[Var[\bar{w}|X]] = \frac{\sigma^2}{n} E(\frac{X^T X}{n})$$

When $\frac{X^T X}{n}$ converges to Q,

$$E[Var[\bar{w}|X]] = 0$$

So $\bar{w}$ converges to $\mathbf{0}(k \times 1)$ vector. Then when N is sufficiently large, $\hat{\beta}$ converges to β. This is the proof of consistency.
There are certain conditions in which the estimators become inconsitent.
1) X is not full rank, or X has multicollinearity 2) $cov[X, \epsilon] \neq 0$

2.3 Efficiency

The least equare estimator has the smallest varaince, and this can be proved by Gauss-Markov theorem.

2.4 Multicollinearity

Suppose we have a regression model that contains two parameters

$$y = \beta_0 + X_1\beta_1 + X_2\beta_2$$

From above, we know variance of $\hat{\beta}$ is

$$Var(\hat{\beta}) = \frac{\sigma^2}{(X^T X)^{-1}}$$

When X only contains 2 variables, $X = (X_1, X_2)$

$$Var(\hat{\beta}_1) = \sigma^2 \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{1}{S_{11}(1 - \frac{S_{12}^2}{S_{11}S_{22}})} = \frac{1}{S_{11}(1 - r_{12}^2)}$$
$$Var(\hat{\beta}_2) = \sigma^2 \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{1}{S_{22}(1 - \frac{S_{12}^2}{S_{11}S_{22}})} = \frac{1}{S_{22}(1 - r_{12}^2)}$$

Where

$$S_{11} = \Sigma(x_{1i} - \hat{x}_1)^2$$
$$S_{22} = \Sigma(x_{2i} - \hat{x}_2)^2$$
$$S_{12} = \Sigma(x_{1i} - \hat{x}_1)(x_{2i} - \hat{x}_2)$$
$$r_{12} = \frac{S_{12}}{\sqrt{S_{11}S_{22}}}$$

r_{12} is the correlation coefficient. In extreme case, when X_1 and X_2 are perfectly correlated, the variance becomes infinite.

3 Model Testing

Lagrange Multiplier(LM) test
Suppose we have two models, one is restriced, the other is unrestricted: Restricted(R): $y = X_1\beta_1 + \epsilon$
Unrestricted (U): $y = X_1\beta_1 + X_2\beta_2 + \epsilon$
Given the unrestricted model, the likehood function is

$$L(\beta_1, \beta_2, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} exp(-\frac{1}{2\sigma^2}(y - X_1\beta_1 - X_2\beta_2)^T(y - X_1\beta_1 - X_2\beta_2))$$

$$S_2 = \frac{\partial L}{\partial \beta_2} = \frac{1}{\sigma^2} X_2^T(y - X_1\beta_1 - X_2\beta_2)$$

When $\beta_2 = 0$, define

$$M_1 = I - X_1(X_1^T X_1)^{-1} X_1^T$$

and $M_1 X_1 = 0$.

$$S_2 = \frac{1}{\sigma^2} X_2^T(y - X\hat{\beta}_1) = \frac{1}{\sigma^2} X_2^T M_1 y = \frac{1}{\sigma^2} X_2^T M_1(X_1\beta_1 + \epsilon) = \frac{1}{\sigma^2} X_2^T M_1 \epsilon$$

The last equal sign uses the fact $M_1 X_1 = 0$.

$$\begin{aligned} Var(X_2^T M_1 \epsilon) &= Var(X_2^T M_1 \epsilon) \\ &= X_2^T M_1 Var(\epsilon)(X_2^T M_1)^T \\ &= X_2^T M_1 M_1^T X_2 Var(\epsilon) \\ &= \sigma^2 X_2^T M_1 X_2 \end{aligned}$$

Define

$$V = X_2^T M_1 X_2$$

Then

$$Var(X_2^T M_1 \epsilon) = \sigma^2 V$$

So $X_2^T M_1 \epsilon$ follows normal distribution with mean 0 and variance $\sigma^2 X_2^T M_1 X_2$. Define

$$Z = \frac{X_2^T M_1 \epsilon}{\sqrt{\sigma^2 X_2^T M_1 X_2}} = \frac{S_2}{\sqrt{\sigma^2 V}}$$

then Z follows standard normal distribution. The **Lagrange Multiplier (LM) test** is defined

$$LM = Z^2 = \frac{(X_2^T M_1 \epsilon)^2}{\sigma^2 X_2^T M_1 X_2} = \frac{S_2^2}{\sigma^2 V}$$

which follows χ^2 distribution with degree of freedom 1.

F test

We define

$$SSE_U = ||y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2||^2$$
$$SSE_R = ||y - X_1\hat{\beta}_1||^2$$

F test is defined as

$$F = \frac{\frac{Extra\ expalined\ variation}{Degree\ of\ Freedom}}{\frac{Remaining\ unexplained\ variation}{Degree\ of\ Freedom}} = \frac{SSE_R - SSE_U}{\frac{SSE_U}{n-1}}$$

Let $X = (X_1, X_2)$, and we define two projection matrices

$$P_U = X(X^TX)^{-1}X^T$$
$$P_R = X_1(X_1^TX_1)^{-1}X_1^T$$

$$SSR_R - SSR_U = y^TP_Ry - y^TP_Uy = y^T(P_R - P_U)y$$

recall

$$\hat{\beta}_2 = (X_2^TM_1X_2)^{-1}X_2^TM_1y$$

The corresponding projection matrix is

$$M_1X_2(X_2^TM_1X_2)^{-1}X_2^TM_1$$

The SSR_R - SSR_U is the additional variance explained by X_2 after removing the linear space of X_1 on X_2. This means the projection matrix corresponding to β_2 is P_U - P_R. So we can get P_U - P_R using the interpretation of projection matrix instead solving for the projection matrix itself.

$$P_U - P_R = M_1X_2(X_2^TM_1X_2)^{-1}X_2^TM_1$$

The extra expained sum of squares by the unrestricted model is

$$SSR_R - SSR_U = y^T(P_R - P_U)y = (X_2^TM_1y)^T(X_2^TM_1X_2)^{-1}X_2^TM_1y$$

with 1 degree of freedom as X_2 only contains 1 parameter.

$$F = \frac{SSR_R - SSR_U}{\frac{SSR_U}{n-k}} = \frac{(X_2M_1y)^T(X_2^TM_1y)}{\hat{\sigma}^2(X_2^TM_1X_2)} = LM$$

We see that F test and LM test are equivalent.

Wald Test

Recall the estimator for β_2 in Eq.4,

$$\hat{\beta}_2 = (X_2^TM_1X_2)^{-1}(X_2^TM_1y)$$

Substitue

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

we get

$$\hat{\beta}_2 = (X_2^T M_1 X_2)^{-1}(X_2^T M_1 \epsilon)$$

Since ϵ $N(0, \sigma^2 I)$, we obtain

$$\beta_2 \sim N(0, \sigma^2 (X_2^T M_1 X_2)^{-1})$$

Thus, scaling by $1/\sigma^2$, we arrive at

$$\frac{1}{\sigma^2} X_2^T M_1 \epsilon \sim N(0, X_2^T M_1 X_2)$$

Construct Wald test W

$$W = \frac{\hat{\beta}_2}{\sqrt{\hat{V}ar(\hat{\beta}_2)}}$$

W follows t distribution. We now show W test is equivalent to LM test. Consider W^2

$$\begin{aligned} W^2 &= \hat{\beta}^T (Var(\hat{\beta}))^{-1} \hat{\beta} = \frac{1}{\sigma^2} \hat{\beta}^T V \hat{\beta} = \frac{1}{\sigma^2} (V^{-1} S_2)^T V V^{-1} S_2 = \frac{1}{\sigma^2} S_2^T V^{-1} V V^{-1} S_2 \\ &= \frac{1}{\sigma^2} S_2^T V^{-1} S_2 \\ &= LM \end{aligned}$$

4 Panel Data Model

We can view panel data as a "two dimensional" data set in which the sample does not only come from different individuals, but also same individual across different time point. We can write the regression model as

$$y_{it} = \alpha_{it} + \sum_k x_{itk} \beta_{itk} + u_{it}$$

where $1 < i < N$, $1 < t < T$, and $1 < k < K$. The equation has total sample size of NT with total number of parameter $NT(K+1)$, therefore it is not estimable. So we will make the following few assumptions

	$\alpha_{it} = \alpha_{is}$	$\alpha_{it} = \alpha_{jt}$	$\beta_{it} = \beta_{is}$	$\beta_{itk} = \beta_{jtk}$
Pooled	yes	yes	yes	yes
Fixed Effect	yes	no	yes	yes
Unrestricted	yes	no	yes	no

4.1 The unrestriced model

$$y_{it} = \alpha_i + \sum_k x_{itk} \beta_{ik} + u_{it}$$

The above equation can be written in matrix form:
for $i = 1$,

$$\begin{pmatrix} y_{11} \\ y_{12} \\ ... \\ y_{1T} \end{pmatrix} = \begin{pmatrix} 1 & x_{111} & x_{112} & ... & x_{11K} \\ 1 & x_{121} & x_{122} & ... & x_{12K} \\ 1 & ... & ... & ... & ... \\ 1 & x_{1T1} & x_{1T2} & ... & x_{1TK} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_{11} \\ \beta_{12} \\ ... \\ \beta_{1K} \end{pmatrix} + \begin{pmatrix} u_{11} \\ u_{12} \\ ... \\ u_{1T} \end{pmatrix}$$

which we can also write as for $i = 2$,

$$\begin{pmatrix} y_{21} \\ y_{22} \\ ... \\ y_{2T} \end{pmatrix} = \begin{pmatrix} 1 & x_{211} & x_{212} & ... & x_{21K} \\ 1 & x_{221} & x_{222} & ... & x_{22K} \\ 1 & ... & ... & ... & ... \\ 1 & x_{2T1} & x_{2T2} & ... & x_{2TK} \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_{21} \\ \beta_{22} \\ ... \\ \beta_{2K} \end{pmatrix} + \begin{pmatrix} u_{21} \\ u_{22} \\ ... \\ u_{2T} \end{pmatrix}$$

So for each i, we can write

$$Y_i = 1_T \alpha_i + X_i \beta_i + U_i$$

where $Y_i = (y_{i1}, y_{i2}, ..., y_{iT})^T$, 1_T is a one vector of length T, X_i is $K \times T$ matrix, $\beta_i = (\beta_{i1}, \beta_{i2}, ..., \beta_{iK})^T$, and $U_i = (u_{i1}, u_{i2}, ..., u_{iT})^T$.

If we consolidate equation set for all the value of i

$$\begin{pmatrix} Y_1 \\ Y_2 \\ ... \\ Y_N \end{pmatrix} = \begin{pmatrix} 1_T & 0 & 0 & ... & 0 \\ 0 & 1_T & 0 & ... & 0 \\ 0 & 0 & 1_T & ... & 0 \\ 0 & 0 & 0 & ... & 1_{NT} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_N \end{pmatrix} + \begin{pmatrix} X_1 & 0 & 0 & ... & 0 \\ 0 & X_2 & 0 & ... & 0 \\ 0 & 0 & X_3 & ... & 0 \\ 0 & 0 & 0 & ... & X_N \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ ... \\ \beta_N \end{pmatrix} + \begin{pmatrix} U_1 \\ U_2 \\ ... \\ U_N \end{pmatrix}$$

To solve for β_i, we can use the strategy of partition regression. β_i is the solution of the the regression

$$MY_i = MX_i \beta + U$$

where

$$M = I - \frac{1}{T} 1_T 1_T^T$$
$$MY_i = y_{it} - \bar{y}_{i.}$$
$$MX_i = x_{it} - \bar{x}_{i.}$$

the estimate of β is

$$\hat{\beta}_i = W_{xx,i}^{-1} W_{xy,i}$$

where

$$W_{xy,i} = \sum_i^T (x_{it} - \bar{x}_{i.})(y_{it} - \bar{y}_{i.})$$
$$W_{xx,i} = \sum_i^T (x_{it} - \bar{x}_{i.})(x_{it} - \bar{x}_{i.})^T$$

4.2 The pooled model

$$y_{it} = \alpha + \sum_k x_{itk}\beta_k + u_{it}$$

The above equation can be written in matrix form:
for $i = 1$,

$$\begin{pmatrix} y_{11} \\ y_{12} \\ ... \\ y_{1T} \\ y_{21} \\ y_{22} \\ ... \\ y_{2T} \\ ... \\ y_{NT} \end{pmatrix} = \begin{pmatrix} 1 & x_{111} & x_{112} & ... & x_{11K} \\ 1 & x_{121} & x_{122} & ... & x_{12K} \\ 1 & ... & ... & ... & ... \\ 1 & x_{1T1} & x_{1T2} & ... & x_{1TK} \\ 1 & x_{211} & x_{212} & ... & x_{21K} \\ 1 & x_{221} & x_{222} & ... & x_{22K} \\ 1 & ... & ... & ... & ... \\ 1 & x_{2T1} & x_{2T2} & ... & x_{2TK} \\ 1 & ... & ... & ... & ... \\ 1 & x_{NT1} & x_{NT2} & ... & x_{NTK} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ ... \\ \beta_K \end{pmatrix} + \begin{pmatrix} u_{11} \\ u_{12} \\ ... \\ u_{1T} \\ u_{21} \\ u_{22} \\ ... \\ u_{2T} \\ ... \\ u_{NT} \end{pmatrix}$$

Similarly, using the solution of β from Eq.5, the estimated β can be written as

$$M = I - \frac{1}{T} 1_T 1_T^T$$

$$\hat{\beta} = W_{xx}^{-1} W_{xy}$$

where

$$W_{xy} = \sum_i^N \sum_t^T (x_{it} - \bar{x}_{..})(y_{it} - \bar{y}_{..})$$
$$W_{xx} = \sum_i^N \sum_t^T (x_{it} - \bar{x}_{..})(x_{it} - \bar{x}_{..})^T$$

4.3 The fixed effect model

$$y_{it} = \alpha_i + \sum_k x_{itk}\beta_k + u_{it}$$

The above equation can be written in matrix form:

$$\begin{pmatrix} y_{11} \\ y_{12} \\ ... \\ y_{1T} \\ y_{21} \\ y_{22} \\ ... \\ y_{2T} \\ ... \\ y_{NT} \end{pmatrix} = \begin{pmatrix} 1_T & 0 & 0 & ... & 0 \\ 0 & 1_T & 0 & ... & 0 \\ 0 & 0 & 1_T & ... & 0 \\ 0 & 0 & 0 & ... & 1_{NT} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_N \end{pmatrix}$$

$$+ \begin{pmatrix} x_{111} & x_{112} & ... & x_{11K} \\ x_{121} & x_{122} & ... & x_{12K} \\ ... & ... & ... & ... \\ x_{1T1} & x_{1T2} & ... & x_{1TK} \\ x_{211} & x_{212} & ... & x_{21K} \\ x_{221} & x_{222} & ... & x_{22K} \\ ... & ... & ... & ... \\ x_{2T1} & x_{2T2} & ... & x_{2TK} \\ ... & ... & ... & ... \\ x_{NT1} & x_{NT2} & ... & x_{NTK} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ ... \\ \beta_K \end{pmatrix} + \begin{pmatrix} u_{11} \\ u_{12} \\ ... \\ u_{1T} \\ u_{21} \\ u_{22} \\ ... \\ u_{2T} \\ ... \\ u_{NT} \end{pmatrix}$$

Similarly, using the solution of β from Eq.5, the M matrix is
the estimated β can be written as

$$\hat{\beta} = W_{xx}^{-1} W_{xy}$$

where

$$W_{xy} = \sum_{i}^{N} \sum_{t}^{T} (x_{it} - \bar{x}_{i.})(y_{it} - \bar{y}_{i.})$$

$$W_{xx} = \sum_{i}^{N} \sum_{t}^{T} (x_{it} - \bar{x}_{i.})(x_{it} - \bar{x}_{i.})^T$$