# 1 ANOVA One-Way Model

One way ANOVA model states as following:

$$
\begin{aligned}
Y_{ij} &= \mu + \alpha_i + \epsilon_{ij} \\
i &\le I; j \le J
\end{aligned}
$$

with the assumption that $\sum_i \alpha_i = 0$, and $\epsilon_{ij}$ follows$N(0, \sigma^2)$.
We now calculate the expectation of the sum square error.

$$
\begin{aligned}
E[SSE] &= E[\sum_{ij}(Y_{ij} - \bar{Y}_{i.})^2] \\
&= E[\sum_{ij}(\mu + \alpha_i + \epsilon_{ij} - (\mu + \alpha_i + \frac{1}{J}\sum_j \epsilon_{ij}))^2] \\
&= E[\sum_{ij}(\epsilon_{ij} - \frac{1}{J}\sum_j \epsilon_{ij})^2] \\
&= \sum_{ij} E[(\epsilon_{ij} - \frac{1}{J}\sum_j \epsilon_{ij})^2] \\
&= \sum_{ij}(E[\epsilon_{ij}^2] - 2E[\epsilon_{ij}][\frac{1}{J}\sum_j \epsilon_{ij}] + E[(\frac{1}{J}\sum_j \epsilon_{ij})^2])
\end{aligned}
$$

Based on the following,

$$
E[\epsilon_{ij}^2] = E[\epsilon_{ij}\epsilon_{kl}] = \sigma^2 \delta_{ik} \delta jl
$$

So the SSE is

$$
\begin{aligned}
E[SSE] &= \sum_{ij}(\sigma^2 - \frac{2}{J}\sigma^2 + \frac{J}{J^2}\sigma^2) \\
&= IJ(^2 - \frac{2}{J}\sigma^2 + \frac{1}{J}\sigma^2) \\
&= IJ\sigma^2 - I\sigma^2 = I(J-1)\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
E[SS\alpha] &= E[\sum_{ij}(\bar{Y}_{i.} - \bar{Y}_{..})^2] \\
&= E[\sum_{ij}(\mu + \alpha_i + \frac{\sum_j \epsilon_{ij}}{J} - (\mu + \sum_i \frac{\alpha_i}{I} + \frac{1}{IJ}\sum_{ij}\epsilon_{ij}))^2] \\
&= E[\sum_{ij}(\alpha_i + \sum_j \frac{\epsilon_{ij}}{J} - \sum_{ij}\frac{\epsilon_{ij}}{IJ})^2] \\
&= J\sum_i \alpha_i^2 + \sum_{ij}(E[(\frac{\sum_j \epsilon_{ij}}{J})^2]) - 2E[\frac{\sum_j \epsilon_{ij}}{J}(\frac{1}{IJ}\sum_{ij}\epsilon_{ij})] + E[(\frac{1}{IJ}\sum_{ij}\epsilon_{ij})^2] \\
&= J\sum_i \alpha_i^2 + \sum_{ij}(\frac{J}{J^2}\sigma^2 - \frac{2J}{IJ^2}\sigma^2 + \frac{IJ}{I^2J^2}\sigma^2) \\
&= J\sum_i \alpha_i^2 + (I-1)\sigma^2
\end{aligned}
$$

$$
\frac{E(SS\alpha)}{I-1} = J\frac{\sum_i \alpha^2}{I-1} + \sigma^2
$$

## 2 ANOVA Two-Way Model

Two way ANOVA model states as following:

$$
\begin{aligned}
Y_{ijk} &= \mu + \alpha_i + \beta_j + \epsilon_{ijk} \\
i &\le I; j \le J; k \le K
\end{aligned}
$$

with the assumption that $\sum_i \alpha_i = 0$, $\sum_j \beta_j = 0$, and $\epsilon_{ijk}$ follows$N(0, \sigma^2)$.
We now calculate the expectation of the sum square error, similar to what we did in section 1.

$$
\begin{aligned}
E[SSE] &= E[\sum_{ijk}(Y_{ijk} - \bar{Y}_{ij.})^2] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \beta_j + \epsilon_{ijk} - (\mu + \alpha_i + \beta_j + \frac{1}{K}\sum_k \epsilon_{ijk}))^2] \\
&= E[\sum_{ijk}(\epsilon_{ijk} - \frac{1}{K}\sum_k \epsilon_{ijk})^2]
\end{aligned}
$$

Based on the SSE we have derived in one-way model, we can easily see

$$
E[SSE] = IJ(K-1)\sigma^2
$$

$$
\begin{aligned}
E[SS\alpha] &= E[\sum_{ijk}(\bar{Y}_{i..} - \bar{Y}_{...})^2] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \frac{\sum_j \beta_j}{J} + \frac{\sum_{jk} \epsilon_{ijk}}{JK} - (\mu + \sum_i \frac{\alpha_i}{I} + \sum_j \frac{\beta_j}{J} + \frac{1}{IJK}\sum_{ijk}\epsilon_{ijk}))^2] \\
&= E[\sum_{ijk}(\alpha_i + \sum_{jk}\frac{\epsilon_{ijk}}{JK} - \sum_{ijk}\frac{\epsilon_{ijk}}{IJK})^2] \\
&= JK\sum_i \alpha_i^2 + (I-1)\sigma^2
\end{aligned}
$$

The last line can be seen based on what we derived in one-way model by replacing j with jk.

# 3 ANOVA Two-Way Nested Model

Two way ANOVA nested model states as following:

$$
\begin{aligned}
Y_{ijk} &= \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk} \\
&i \leq I; j \leq J; k \leq K
\end{aligned}
$$

with the assumption that $\sum_i \alpha_i = 0$, $\sum_j \beta_j(i) = 0$, and $\epsilon_{ijk}$ follows$N(0, \sigma^2)$. We now calculate the expectation of the sum square error, similar to what we did in section 2.

$$
\begin{aligned}
E[SSE] &= E[\sum_{ijk}(Y_{ijk} - \bar{Y}_{ij.})^2] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk} - (\mu + \alpha_i + \beta_{j(i)} + \frac{1}{K}\sum_k \epsilon_{ijk}))^2] \\
&= E[\sum_{ijk}(\epsilon_{ijk} - \frac{1}{K}\sum_k \epsilon_{ijk})^2]
\end{aligned}
$$

Based on the SSE we have derived in one-way model, by replacing j in one-way model with jk, we can easily see

$$
E[SSE] = IJ(K-1)\sigma^2
$$

$$
\begin{aligned}
E[SS\beta|\alpha] &= E[\sum_{ijk}(\bar{Y}_{ij.} - \bar{Y}_{...})^2] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \beta_{j(i)} + \sum_k \frac{\epsilon_{ijk}}{K} - (\mu + \sum_i \alpha_i + \frac{\sum_j \beta_{j(i)}}{J} + \frac{1}{IJK}\sum_{ijk}\epsilon_{ijk}))^2] \\
&= K\sum_{ij}\beta_{j(i)}^2 + \sum_{ijk} E[(\sum_k \frac{\epsilon_{ijk}}{K} - \frac{1}{IJK}\epsilon_{ijk})^2] \\
&= K\sum_{ij}\beta_{j(i)}^2 + \sum_{ijk}(\frac{K}{K^2} - 2\frac{1}{IJK^2}K\sigma^2 + \frac{1}{I^2J^2K^2}IJK\sigma^2) \\
&= K\sum_{ij}\beta_{j(i)}^2 + (IJ-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.

$$
\begin{aligned}
E[SS\alpha] &= E[\sum_{ijk}(\bar{Y}_{i..} - \bar{Y}_{...})^2] \\
&= E[\sum_{ijk}(\mu + \alpha_i + \frac{\sum_j \beta_{j(i)}}{J} + \frac{\sum_{ijk}\epsilon_{ijk}}{JK} - (\mu + \sum_i \frac{\alpha_i}{I} + \frac{\sum_j \beta_{j(i)}}{J} + \frac{1}{IJK}\sum_{ijk}\epsilon_{ijk}))^2] \\
&= E[\sum_{ijk}(\alpha_i + \sum_{jk}\frac{\epsilon_{ijk}}{JK} - \sum_{ijk}\frac{\epsilon_{ijk}}{IJK})^2] \\
&= JK\sum_i \alpha_i^2 + (I-1)\sigma^2
\end{aligned}
$$

The last line is based on what we derived in one-way model.